

Integrability Condition in the Statistical Model and the Addition Formula of $g = 2$ Hyperelliptic Function

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Abstract

The integrability condition of the Ising model is understood as the $SU(2)$ integrability condition and also as the model parameterized by the elliptic function, where the integrability condition is understood as the addition theorem of the elliptic function. The generalization of this integrability condition is to find the solution of the higher rank Lie group integrability condition and also to find the model parameterized by higher genus hyperelliptic function. For the preparation of this purpose, we give the explicit formula of the addition formula for $g = 2$ hyperelliptic function.

Keywords:	Ising model ,	integrability condition ,	theta function
	hyperelliptic function,	addition formula,	Lie group

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1 Introduction

There are many two dimensional integrable statistical models [1]. The fundamental integrable statistical model is the Ising model [2], which is parameterized by the elliptic function to satisfy the integrable condition. The general Yang-Baxter equation, which has the difference property, for the spin model is given in the form

$$U(x)V(x+y)U(y) = V(y)U(x+y)V(x). \quad (1.1)$$

This type of Yang-Baxter equation is called the integrability condition, because the Yang-Baxter equation of this type says that the product of three group actions for two different paths gives the same group action, that is, the product of three group action is path independent. In general, if we start from some fixed point in the group and multiplying the group action through certain path to the final point in the group, the total group action is path independent in the above integrable model.

This integrable model seems to have the connection with the algebraic function. The algebraic function has the addition formula. The addition formula of the trigonometry function can be written as the $U(1)$ integrable condition in the form Eq.(1.1) through the Euler's relation $U(x) = \exp(ix) = \cos(x) + i\sin(x)$, $V(x) = 1$. The addition formula of the elliptic function can be written as the $SU(2)$ Lie group integrable condition in the form Eq.(1.1) [3, 4, 5] with

$$U(x) = \exp\{iam(x, k)J_z\}, \quad V(x) = \exp\{iam(kx, 1/k)J_x\}, \quad (1.2)$$

where J_x, J_z are Lie group elements of $SO(3) \cong SU(2)/Z_2$. We must notice that we can write the integrable condition in the Lie group form by using the elliptic function with some moduli and the elliptic function with the dual moduli, that is, the Lie group structure and the discrete moduli transformation is mutually connected.

There are two directions to generalize the above integrable model. One direction of the generalization is to consider the two variable integrable condition with difference property in the form

$$U(x_1, y_1)V(x_1 + x_2, y_1 + y_2)U(x_2, y_2) = V(x_2, y_2)U(x_1 + x_2, y_1 + y_2)V(x_1, y_1). \quad (1.3)$$

Another direction of the generalization is to consider the higher genus algebraic function, $g = 2$ hyperelliptic function, which has two arguments.

Then it is quite desirable that the addition formula of $g = 2$ hyperelliptic function can be written in the integrable condition in the form Eq(1.3).

If this expectation is true, the integrable condition (path independence) of the algebraic function on the Riemann surface, which is nothing but the addition formula of the hyperelliptic function, is expected to be written as the integrable condition of certain Lie group.

For that preparation, we explicitly derive the addition formula for $g = 2$ hyperelliptic function in this paper.

2 Addition Relation of $g = 2$ Theta Function

The addition relation of $g = 2$ hyperelliptic theta function is first given by Rosenhain[6, 7], and Königsberger[8] give the fundamental relation to make the addition theorem.

The explicit addition formula is given by Kossak[9] but he give only the sketch to derive the addition formula and did not give all the addition formula. Then we will give the detailed derivation and give all addition formula.

The theta function with two variables is defined by

$$\begin{aligned} & \vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u, v; \tau_1, \tau_2, \tau_{12}) \\ &= \sum_{m, n \in \mathbb{Z}} \exp \left\{ \pi i \left(\tau_1 \left(m + \frac{a}{2} \right)^2 + \tau_2 \left(n + \frac{c}{2} \right)^2 + 2\tau_{12} \left(m + \frac{a}{2} \right) \left(n + \frac{c}{2} \right) \right) \right. \\ & \quad \left. + 2\pi i \left(\left(m + \frac{a}{2} \right) \left(u + \frac{b}{2} \right) + \left(n + \frac{c}{2} \right) \left(v + \frac{d}{2} \right) \right) \right\}, \end{aligned} \quad (2.1)$$

where we assume that $\text{Im}\tau_1 > 0$, $\text{Im}\tau_2 > 0$, $(\text{Im}\tau_1)(\text{Im}\tau_2) - (\text{Im}\tau_{12})^2 > 0$ in order that the summation of $m, n \in \mathbb{Z}$ becomes convergent. We can rename $m \rightarrow m$, $n \rightarrow -n$, so that we can always choose $\text{Im}\tau_{12} > 0$ so we assume $\text{Im}\tau_{12} > 0$.

The Riemann's theta relation in this case is given by

$$\begin{aligned} & \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (u_i, v_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (u_i, v_i) \\ &= \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i) + \prod_{i=1}^4 \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (\tilde{u}_i, \tilde{v}_i), \end{aligned} \quad (2.2)$$

where

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \end{pmatrix} = \begin{pmatrix} (u_1 + u_2 + u_3 + u_4)/2 \\ (u_1 + u_2 - u_3 - u_4)/2 \\ (u_1 - u_2 + u_3 - u_4)/2 \\ (u_1 - u_2 - u_3 + u_4)/2 \end{pmatrix}, \quad \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{v}_3 \\ \tilde{v}_4 \end{pmatrix} = \begin{pmatrix} (v_1 + v_2 + v_3 + v_4)/2 \\ (v_1 + v_2 - v_3 - v_4)/2 \\ (v_1 - v_2 + v_3 - v_4)/2 \\ (v_1 - v_2 - v_3 + v_4)/2 \end{pmatrix}. \quad (2.3)$$

We parameterize in the following way

$$\begin{aligned} & \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha', \beta') \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y - y' + \alpha', z - z' + \beta') \\ &+ \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (\alpha', \beta') \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y - y' + \alpha', z - z' + \beta') \\ &+ \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\alpha', \beta') \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y - y' + \alpha', z - z' + \beta') \\ &+ \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (\alpha', \beta') \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y - y' + \alpha', z - z' + \beta') \\ &= \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha + \alpha', z + \beta + \beta') \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha - \alpha', z' + \beta - \beta') \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\ &+ \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + \alpha + \alpha', z + \beta + \beta') \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y' + \alpha - \alpha', z' + \beta - \beta') \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \end{aligned}$$

$$\begin{aligned} & +\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\ & -\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \end{aligned} \quad (2.7)$$

for such α, β as $\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) = 0$.

Addition Formula 1):

Putting $\alpha = 1/2$, $\beta = 1/2$ in Eq.(2.5), we have

$$\begin{aligned}
& \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y + y', z + z') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y - y', z - z') \\
&= \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta^2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') - \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') + \vartheta^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z'). \tag{2.8}
\end{aligned}$$

Addition Formula 2):

Putting $\alpha = 0$, $\beta = 1/2$ in Eq.(2.5), we have

$$\begin{aligned}
& \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y+y', z+z') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y-y', z-z') \\
&= \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\
&+ \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y', z') .
\end{aligned} \tag{2.9}$$

Addition Formula 3):

Putting $\alpha = 1/2$, $\beta = 0$ in Eq.(2.5), we have

$$\begin{aligned}
& \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y+y', z+z') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y-y', z-z') \\
&= \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
&+ \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z'). \tag{2.10}
\end{aligned}$$

we have

$$\begin{aligned}
& \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0,0) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y+y', z+z') \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y-y', z-z') \\
&= \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y',z') \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y',z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y',z') \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y',z') \\
&+ \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y',z') \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y',z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y,z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y',z') \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y',z'). \tag{2.23}
\end{aligned}$$

We have numerically checked these relations Eq.(2.8) \sim Eq.(2.23) by REDUCE.

3 Addition Formula of $g = 2$ Hyperelliptic Function

We define the hyperelliptic function in the form

$$F \begin{bmatrix} a & c \\ b & d \end{bmatrix} (y, z) = \vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (y, z) / \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z). \quad (3.1)$$

Addition Formula 1):

Taking the ratio of Eq.(2.9)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_1}{B_0 B_1}, \quad (3.2)$$

where

$$\begin{aligned} A_1 &= F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\ &+ F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y', z'), \\ B_0 &= 1 - F^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') - F^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F^2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \\ &+ F^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z'), \\ B_1 &= F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0). \end{aligned}$$

Addition Formula 2):

Taking the ratio of Eq.(2.10)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_2}{B_0 B_2}, \quad (3.3)$$

where

$$\begin{aligned} A_2 &= F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\ &+ F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z'), \\ B_2 &= F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0). \end{aligned}$$

Addition Formula 3):

Taking the ratio of Eq.(2.11)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_3}{B_0 B_3}, \quad (3.4)$$

where

$$\begin{aligned} A_3 &= F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\ &+ F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z'), \\ B_3 &= F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0). \end{aligned}$$

Addition Formula 4):

Taking the ratio of Eq.(2.12)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_4}{B_0 B_4}, \quad (3.5)$$

where

$$\begin{aligned} A_4 &= F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\ &+ F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z'), \\ B_4 &= F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0). \end{aligned}$$

Addition Formula 5):

Taking the ratio of Eq.(2.13)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_5}{B_0 B_5}, \quad (3.6)$$

where

$$\begin{aligned} A_5 &= F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \end{aligned}$$

$$\begin{aligned}
& -F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\
& + F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z'), \\
& B_5 = F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 6):

Taking the ratio of Eq.(2.14)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_6}{B_0 B_6}, \quad (3.7)$$

where

$$\begin{aligned}
A_6 &= F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\
& - F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\
& + F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\
& - F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z'), \\
B_6 &= F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 7):

Taking the ratio of Eq.(2.15)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_7}{B_0 B_7}, \quad (3.8)$$

where

$$\begin{aligned}
A_7 &= F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
& - F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
& + F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\
& - F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z'), \\
B_7 &= F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 8):

Taking the ratio of Eq.(2.16)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_8}{B_0 B_8}, \quad (3.9)$$

where

$$\begin{aligned}
A_8 &= F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\
&+ F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\
&- F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z'), \\
B_8 &= F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 9):

Taking the ratio of Eq.(2.17)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_9}{B_0 B_9}, \quad (3.10)$$

where

$$\begin{aligned}
A_9 &= F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\
&- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\
&- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\
&+ F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z'), \\
B_9 &= F \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 10):

Taking the ratio of Eq.(2.18)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_{10}}{B_0 B_{10}}, \quad (3.11)$$

where

$$\begin{aligned}
A_{10} &= F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\
&- F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \\
&+ F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y', z'), \\
B_{10} &= F \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 11):

Taking the ratio of Eq.(2.19)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_{11}}{B_0 B_{11}}, \quad (3.12)$$

where

$$\begin{aligned} A_{11} &= F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\ &+ F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z'), \\ B_{11} &= F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (0, 0) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0). \end{aligned}$$

Addition Formula 12):

Taking the ratio of Eq.(2.20)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y + y', z + z') = \frac{A_{12}}{B_0 B_{12}}, \quad (3.13)$$

where

$$\begin{aligned} A_{12} &= F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\ &+ F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z'), \\ B_{12} &= F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) F \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0). \end{aligned}$$

Addition Formula 13):

Taking the ratio of Eq.(2.21)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_{13}}{B_0 B_{13}}, \quad (3.14)$$

where

$$\begin{aligned} A_{13} &= F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\ &- F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \end{aligned}$$

$$\begin{aligned}
& +F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\
& -F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} (y', z'), \\
& B_{13} = F \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (0, 0) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 14):

Taking the ratio of Eq.(2.22)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_{14}}{B_0 B_{14}}, \quad (3.15)$$

where

$$\begin{aligned}
A_{14} &= F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
& -F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
& +F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \\
& -F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z'), \\
& B_{14} = F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

Addition Formula 15):

Taking the ratio of Eq.(2.23)/Eq.(2.8), we have the addition formula

$$F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y + y', z + z') = \frac{A_{15}}{B_0 B_{15}}, \quad (3.16)$$

where

$$\begin{aligned}
A_{15} &= F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
& -F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
& +F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \\
& -F \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) F \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') F \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z'), \\
& B_{15} = F \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (0, 0) F \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0).
\end{aligned}$$

We have numerically checked these addition formula Eq.(3.2) ~ Eq.(3.16) by REDUCE.

4 Summary and Discussion

The integrability condition of the Ising model is understood as the $SU(2)$ integrability condition and also as the model parameterized by the elliptic function, where the integrability condition is understood as the addition theorem of the elliptic function. The generalization of this integrability condition is to find the solution of the higher rank Lie group integrability condition and also to find the model parameterized by higher genus hyperelliptic function. For the preparation of this purpose, we give the explicit formula of the addition formula for $g = 2$ hyperelliptic function.

The trivial case is the $SO(4) \cong [SU(2) \otimes SU(2)]/Z_2$ integrable condition in the form

$$U(x_1, y_1)V(x_1 + x_2, y_1 + y_2)U(x_2, y_2) = V(x_2, y_2)U(x_1 + x_2, y_1 + y_2)V(x_1, y_1), \quad (4.1)$$

where

$$U(x, y) = \tilde{U}(x, k_1) \otimes \tilde{U}(y, k_2), \quad V(x, y) = \tilde{V}(x, k_1) \otimes \tilde{V}(y, k_2), \quad (4.2)$$

and

$$\begin{aligned} \tilde{U}(x, k_1) &= \exp\{i\text{am}(x, k_1)J_z\}, & \tilde{U}(y, k_2) &= \exp\{i\text{am}(y, k_2)J_z\}, \\ \tilde{V}(x, k_1) &= \exp\{i\text{am}(k_1x, 1/k_1)J_x\}, & \tilde{V}(y, k_2) &= \exp\{i\text{am}(k_2y, 1/k_2)J_x\}. \end{aligned}$$

The algebraic function with non-trivial two argument is $g = 2$ hyperelliptic function instead of the direct product of the elliptic function. And we expect that the Lie group integrability condition will be written as the form of the $SU(4)$ Lie group integrable condition. Because we have 15 hyperelliptic functions in $g = 2$ case and all these 15 hyperelliptic functions are mutually connected in the addition formula, we expect that the number of the hyperelliptic functions is equal to the number of generators of the Lie group. Then we expect that the Lie group which is connected with $g = 2$ hyperelliptic functions is $SU(4)$ because the number of generators of $SU(4)$ is 15.

We further expect that the addition formula for genus g hyperelliptic function will be written as the integrable condition of the $SU(2^g)$ Lie group, because the number of the hyperelliptic functions of genus g is equal to $2^{2g} - 1$ which is equal to the number of generators of $SU(2^g)$. Then the Jacobian varieties will have the $SU(2^g)$ Lie group structure through the addition formula of the theta function.

As the $K3$ surface can be parameterized by the $g = 2$ hyperelliptic theta function[10], some special algebraic varieties may have the Lie group structure.

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$$\begin{aligned}
& +\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y + \alpha, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\
& -\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y + \alpha, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z').
\end{aligned}
\tag{A.5}$$

The left-hand side of Eq.(A.4) and Eq.(A.5) is equal, so that we have the relation that the right-hand side of Eq.(A.4) and Eq.(A.5) is equal. Further, we replace $\alpha \rightarrow \alpha + 1/2$. Then we have the relation

[illegible]

Adding Eq.(A.3) and Eq.(A.6), we have

$$\begin{aligned}
& \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y - y', z - z') \\
&= \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \\
&+ \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z'). \tag{A.7}
\end{aligned}$$

We further replace $y \rightarrow y + 1/2$, $z \rightarrow z + 1/2$, $y' \rightarrow y' + 1/2$, $z' \rightarrow z' + 1/2$, and we have

$$\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y - y', z - z')$$

$$\begin{aligned}
&= \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\
&+ \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\
&= \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \\
&- \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \\
&+ \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z'). \quad (\text{A.8})
\end{aligned}$$

In the above, we use the following relation. First we notice $\vartheta \begin{bmatrix} a & c \\ b & d \end{bmatrix} (u, v; \tau_1, \tau_2, \tau_{12}) = \vartheta \begin{bmatrix} c & a \\ d & b \end{bmatrix} (v, u; \tau_2, \tau_1, \tau_{12})$, then the left-hand side of Eq.(A.7) does not change under the rename of $\alpha \leftrightarrow \beta$, $y \leftrightarrow z$, $y' \leftrightarrow z'$, $\tau_1 \leftrightarrow \tau_2$, then the first column and the second column exchanged expression of the theta function in the right-hand side of Eq.(A.7) is equal to the original expression. This Eq.(A.8) is the Kossak's Fundamental relation 1) of Eq.(2.5).

Kossak's Fundamental Relation 2):

We choose α, β (6 cases) in such a way as $\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) = 0$. Then Eq.(A.1) and Eq.(A.2) is given by

$$\begin{aligned}
& \left(\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{2}, \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y - y' + \frac{1}{2}, z - z' + \frac{1}{2}) \right) \\
&= \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
&+ \left(\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
&= - \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \right) \\
&+ \left(\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \right)
\end{aligned}$$

$$\begin{aligned}
& + \left(\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \right) \\
& - \left(\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \right).
\end{aligned} \tag{A.9}$$

As the second term and the third term is equal, we have

$$\begin{aligned}
0 = & \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
& - \left(\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
& - \left(\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
& + \left(\vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
& + \left(\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \right) \\
& - \left(\vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \right) \\
& - \left(\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \right) \\
& + \left(\vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \left(y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \right).
\end{aligned} \tag{A.10}$$

where we use the relation that the left-hand side of Eq.(A.9) does not change under the rename of $\alpha \leftrightarrow \beta$, $y \leftrightarrow z$, $y' \leftrightarrow z'$, $\tau_1 \leftrightarrow \tau_2$, then the first column and the second column exchanged expression of the theta function in the right-hand side Eq.(A.9) is equal to the original expression.

Another Relation

Next we make another relation. Then we replace $y \rightarrow y + 1/2$, $y' \rightarrow y' + 1/2$ in Eq.(A.4), and we have

$$\begin{aligned}
& \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \left(0, \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y - y', z - z' + \frac{1}{2}) \\
& - \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(0, \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y - y', z - z' + \frac{1}{2}) \\
& = \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(y + \alpha, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \\
& - \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(y + \alpha, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \\
& + \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \left(y + \alpha, z + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \left(y' + \alpha, z' + \beta + \frac{1}{2} \right) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z')
\end{aligned}$$

$$-\vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} (y', z'). \quad (\text{A.11})$$

Adding Eq.(A.4) and Eq.(A.11), and further replace $\alpha \rightarrow \alpha + 1/2$, we have

[illegible]

Then we add Eq.(A.10) and (Eq.(A.12)).

$$\begin{aligned}
& \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (\alpha + \frac{1}{2}, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (0, \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha + \frac{1}{2}, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y - y', z - z' + \frac{1}{2}) \right) \\
&= \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y', z') \right) \\
&+ \left(\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y + \alpha + \frac{1}{2}, z + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y' + \alpha + \frac{1}{2}, z' + \beta + \frac{1}{2}) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y', z') \right).
\end{aligned}
\tag{A.13}$$

By replacing $y \rightarrow y + 1$, $z \rightarrow z + 1/2$, $y' \rightarrow y' + 1/2$, $z' \rightarrow z' + 1/2$, we have

$$\left(\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y - y', z - z')\right)$$

$$\begin{aligned}
&= \left(\vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y', z') \right) \\
&+ \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} (y', z') \right). \quad (\text{A.14})
\end{aligned}$$

This is the Kossak's Fundamental Relation 2) of Eq.(2.6).

Kossak's Fundamental Relation 3):

From the above, we also have the Kossak's Fundamental relation 3) of Eq.(2.7) in the form

$$\begin{aligned}
&\left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (\alpha, \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (0, 0) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + y' + \alpha, z + z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y - y', z - z') \right) \\
&= \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y', z') \right) \\
&+ \left(\vartheta \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \right) \\
&- \left(\vartheta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} (y + \alpha, z + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} (y, z) \vartheta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (y' + \alpha, z' + \beta) \vartheta \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} (y', z') \right), \quad (\text{A.15})
\end{aligned}$$

where we use the relation that the left-hand side of Eq.(A.14) does not change under the rename of $\alpha \leftrightarrow \beta$, $y \leftrightarrow z$, $y' \leftrightarrow z'$, $\tau_1 \leftrightarrow \tau_2$, then the first column and the second column exchanged expression of the theta function in the right-hand side of Eq.(A.14) is equal to the original expression.